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Lie symmetries and infinite-dimensional Lie algebras of certain nonlinear dissipative systems

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Abstract. We point out that the dissipative force-free Duffing oscillator and Holmes–Rand nonlinear oscillator admit infinite dimensional symmetry algebras for certain parametric choices for which the systems become integrable. From the nature of the symmetry vector fields one can also write down the integrals of motion for the above systems. In addition we point out that the other dissipative systems discussed in the literature also admit infinite dimensional Lie algebras.

1. Introduction

An important problem in the study of nonlinear dynamical systems is to find under what conditions a given dynamical system is integrable or not. In order to identify the integrable parameters of the given system, particularly Hamiltonian ones, three different techniques have widely been used, namely Painlevé analysis, Lie symmetry analysis and direct method of finding involutive integrals of motion [1–3]. Among them, the group theoretical method is of special significance. In particular the study of generalized Lie symmetries of nonlinear Hamiltonian systems has attracted much attention in recent years [4] because it not only gives the integrable parameters along with the integrals of motion but also gives the separable coordinates whenever they exist [5].

As far as nonlinear non-Hamiltonian systems are concerned much less is known even though some techniques have been proposed to find the integrable cases of such systems [2, 6, 7]. Here also the study of Lie symmetries plays a prominent role because once again it not only gives the integrable parameters of the problem but also gives the associated integrals of motion in a straightforward fashion. Recently the integrability properties of some of the nonlinear dissipative systems such as the Lorenz model, two-dimensional Lotka–Volterra equation and three-wave interaction problem have been studied through Lie symmetry analysis [8–10]. In this paper we wish to investigate the invariance and integrability properties of two physically interesting dissipative nonlinear systems, namely the force-free Duffing oscillator and the Holmes–Rand nonlinear oscillator for which symmetry analysis has not been performed before as far as our knowledge goes. Our motivation to study the above systems is twofold. Our first aim is, apart from finding the Lie symmetries and its associated integrals of motion of a dynamical system in a conventional way, to show that one can generate an infinite sequence of symmetries and infinite dimensional Lie algebra from the basic vector fields of lower-order symmetries which have already been found.

Our second goal is to bring out the unexplored invariance properties of the Duffing and Holmes–Rand oscillators through Lie symmetry analysis. For the force-free Duffing case,

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we show that an infinite number of symmetries exist exactly for the parametric choice for which the Painlevé property is found to hold and explicitly integrable. On the other hand, for the Holmes–Rand oscillator, even though it admits movable algebraic branch points [11], we point out through our Lie symmetry analysis that there exists a parametric choice admitting an infinite dimensional symmetry algebra and integral of motion.

We have also shown that similar infinite dimensional Lie algebras exist for other nonlinear dissipative systems discussed in the literature such as the two-dimensional Lotka–Volterra equation, three-wave interaction problem and the Lorenz system. For example Baumann and Freyberger [9] have shown that the Lotka–Volterra equation admits a three-parameter symmetry group (up to quadratic power in the variable y) for a specific parametric choice. However, on closer examination we have found a fourth vector field S_4, which is also of degree two in the variable y, exists. These four vector fields are then found to give rise to an infinite dimensional symmetry algebra as in the previous cases. Similarly to the case of the three-wave interaction problem, it is completely integrable for only one parametric choice and partially integrable for four other parametric choices [10]. In the completely integrable case we find that the system admits a six-parameter group and in the partially integrable case it admits only a two-parameter group (up to quadratic power in the third variable z). Now constructing the commutator algebra between the six vector fields in the completely integrable case we have shown that one can generate an infinite sequence of symmetries and functionally dependent integrals of motion without explicitly solving the invariance condition. Recently the Lie symmetries and the associated integrals of motion of the Lorenz system have been investigated by Sen and Tabor [8]. For the completely integrable case they have reported that the system admits a four-parameter group of symmetries and in the partially integrable cases it admits two parameter groups (up to cubic power in the variable y). In the completely integrable case, we find that there are two more vector fields compatible with the above ansatz and by including these two vector fields we have shown that one can generate higher-order symmetries and an infinite dimensional Lie algebra.

The plan of the paper is as follows. In appendix A we briefly summarize the invariance condition (used in the text) for a given set of coupled first-order ordinary differential equations under a one-parameter Lie group of transformations. In section 2 we show that the force-free Duffing oscillator admits a four-parameter Lie symmetry group (up to cubic power in the momentum variable y) by solving the invariance condition. Then by constructing the commutator algebra between the four vector fields a new vector field which is the fifth power in the variable y is obtained. Now including the new vector field with the previous four vector fields and constructing the commutator algebra between them again we find that one can generate new vector fields of higher-order symmetries and higher-order functionally dependent integrals of motion without solving the invariance condition. This procedure can be continued *ad infinitum*, thereby generating an infinite dimensional Lie algebra. In section 3 we report the Lie symmetries and integrals of motion for the Holmes–Rand nonlinear oscillator. In this case also we have generated an infinite dimensional Lie algebra for a specific choice of the parameters. In section 4 we briefly point out how the other nonlinear dissipative systems discussed in the literature such as the two-dimensional Lotka–Volterra equation, the three-wave interaction problem and the Lorenz system admit infinite dimensional Lie algebras in the completely integrable cases. In section 5 we present our conclusions.

2. Duffing oscillator

The equation of motion of the force-free Duffing oscillator is

$$\ddot{x} + c_1\dot{x} + c_2x + x^3 = 0 \tag{2.1}$$

where c_1 and c_2 are parameters and the dot denotes differentiation with respect to time. System (2.1) is a ubiquitous model which arises in many branches of science and engineering such as the study of oscillations of a pendulum, oscillations of a buckled beam and so on [12].

2.1. Lie symmetries

Equation (2.1) can be written as the following two first-order coupled ordinary differential equations:

$$\begin{aligned} \Delta : \dot{x} &= y \\ \dot{y} &= -(c_1y + c_2x + x^3). \end{aligned} \tag{2.2}$$

Applying the invariance condition (A.6) to the system (2.2) we get

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= -[\eta_1(c_2 + 3x^2) + c_1\eta_2]. \end{aligned} \tag{2.3}$$

As equation (2.3) cannot be explicitly solved, we make the ansatz that η_1 and η_2 are polynomial in the variable y. In fact, we begin with a linear form in y, then proceed to a quadratic form and finally we assume a cubic form in y from which we succeed in identifying a non-trivial infinite dimensional Lie algebra of symmetry vector fields which can be associated directly with the integral of motion for a suitable parametric choice. For simplicity we present only the calculations for the cubic form, by assuming

$$\begin{aligned} \eta_1 &= a_1 + a_2y + a_3y^2 + a_4y^3 \\ \eta_2 &= b_1 + b_2y + b_3y^2 + b_4y^3 \end{aligned} \tag{2.4}$$

where the a_i's and b_i's, $i = 1, 2, 3, 4$ are functions of t and x alone.

Substituting (2.4) in (2.3) and equating various powers of y we get the following set of linear partial differential equations.

$$a_{4x} = 0 \qquad b_{4x} = 0 \tag{2.5a}$$

$$a_{4t} + a_{3x} - 3c_1a_4 - b_4 = 0 \qquad b_{4t} + b_{3x} - 2c_1b_4 + (c_2 + 3x^2)a_4 = 0 \tag{2.5b}$$

$$\begin{aligned} &a_{3t} + a_{2x} - 2c_1a_3 - 3(c_2x + x^3)a_4 - b_3 = 0 \\ &b_{3t} + b_{2x} - 3(c_2x + x^3)b_4 - c_1b_3 + (c_2 + 3x^2)a_3 = 0 \end{aligned} \tag{2.5c}$$

$$\begin{aligned} &a_{2t} + a_{1x} - c_1a_2 - 2(c_2x + x^3)a_3 - b_2 = 0 \\ &b_{2t} + b_{1x} - 2(c_2x + x^3)b_3 + (c_2 + 3x^2)a_2 = 0 \end{aligned} \tag{2.5d}$$

$$a_{1t} - a_2(c_2x + x^3) - b_1 = 0 \qquad b_{1t} - b_2(c_2x + x^3) + c_1b_1 + (c_2 + 3x^2)a_1 = 0. \tag{2.5e}$$

Solving (2.5*a*)–(2.5*e*) we get

$$a_4 = a_{41}(t) \tag{2.6a}$$

$$a_3 = [b_{41} + 3c_1 a_{41} - \dot{a}_{41}]x + a_{31}(t) \tag{2.6b}$$

$$a_2 = a_{41}\left(\frac{x^4}{2}\right) + [4c_1 b_{41} + 2c_2 a_{41} - 2\dot{b}_{41} + 6c_1^2 a_{41} - 5c_1\dot{a}_{41} + \ddot{a}_{41}]\left(\frac{x^2}{2}\right)$$
$$+ [b_{31} + 2c_1 a_{31} - \dot{a}_{31}]x + a_{21}(t) \tag{2.6c}$$

$$a_1 = [-\tfrac{3}{2}\dot{a}_{41} + 4c_1 a_{41} + 2b_{41}]\left(\frac{x^5}{5}\right) + a_{31}\left(\frac{x^4}{4}\right)$$
$$+ [-\tfrac{1}{2}\dddot{a}_{41} + 3c_1\ddot{a}_{41} - (2c_2 + \tfrac{11}{2}c_1^2)\dot{a}_{41} + (5c_1c_2 + 3c_1^3)a_{41} + \tfrac{3}{2}\ddot{b}_{41}$$
$$- \tfrac{9}{2}c_1\dot{b}_{41} + (3c_1^2 + 3c_2)b_{41}]\left(\frac{x^3}{3}\right)$$
$$+ [\ddot{a}_{31} - 3c_1\dot{a}_{31} + (2c_1^2 + c_2)a_{31} - 2\dot{b}_{31} + 2c_1 b_{31}]\left(\frac{x^2}{2}\right)$$
$$+ [b_{21} + c_1 a_{21} - \dot{a}_{21}]x + a_{11}(t) \tag{2.6d}$$

$$b_4 = b_{41}(t) \tag{2.6e}$$

$$b_3 = -a_{41}x^3 + [2c_1 b_{41} - c_2 a_{41} - \dot{b}_{41}]x + b_{31}(t) \tag{2.6f}$$

$$b_2 = [4\dot{a}_{41} - 10c_1 a_{41}]\left(\frac{x^4}{4}\right) - a_{31}x^3 + [\ddot{b}_{41} - 3c_1\dot{b}_{41} + (2c_2 + 2c_1^2)b_{41} + 2c_2\dot{a}_{41} - 4c_1c_2a_{41}]\left(\frac{x^2}{2}\right)$$
$$+ [c_1 b_{31} - c_2 a_{31} - \dot{b}_{31}]x + b_{21}(t) \tag{2.6g}$$

$$b_1 = -\tfrac{7}{2}a_{41}\left(\frac{x^7}{7}\right) + [-\tfrac{5}{2}\ddot{a}_{41} + 10c_1\dot{a}_{41} - (\tfrac{15}{2}c_2 + 9c_1^2)a_{41} + \dot{b}_{41} - 2c_1 b_{41}]\left(\frac{x^5}{5}\right)$$
$$+ [4\dot{a}_{31} - 6c_1 a_{31} - b_{31}]\left(\frac{x^4}{4}\right) + [-\tfrac{3}{2}c_2\ddot{a}_{41} + (9/2)c_1c_2\dot{a}_{41} - 3c_1^2c_2a_{41}$$
$$- \tfrac{1}{2}\dddot{b}_{41} + \tfrac{3}{2}c_1\ddot{b}_{41} - (c_1^2 + 2c_2)\dot{b}_{41} + 2c_1c_2b_{41} - 3c_2^2a_{41} - 3a_{21}]\left(\frac{x^3}{3}\right)$$
$$+ [\ddot{b}_{31} - c_1\dot{b}_{31} + 2c_2\dot{a}_{31} - 2c_1c_2a_{31} + c_2b_{31}]\left(\frac{x^2}{2}\right) - [c_2 a_{21} + \dot{b}_{21}]x + b_{11}(t) \tag{2.6h}$$

where a_{41}, a_{31}, a_{21}, a_{11}, b_{41}, b_{31}, b_{21} and b_{11} are functions of time and the dot denotes differentiation with respect to time. Substituting (2.6*a*–*h*) in (2.5*e*) and equating various powers of x we get a set of linear differential equations involving the functions a_{41}, a_{31}, a_{21}, a_{11}, b_{41}, b_{31}, b_{21} and b_{11}. Solving them we get non-trivial forms for the above functions only for the parametric choice

$$2c_1^2 = 9c_2. \tag{2.7}$$

For this choice the associated functions become

$$
\begin{aligned}
&a_{11} = 0 \qquad b_{11} = 0 \\
&a_{21} = A + B\exp[\tfrac{1}{3}c_1t] \qquad b_{21} = -Ac_1 - \tfrac{1}{3}c_1B\exp[\tfrac{1}{3}c_1t] \\
&a_{31} = 0 \qquad b_{31} = 0 \\
&a_{41} = C\exp[\tfrac{5}{3}c_1t] + D\exp[\tfrac{4}{3}c_1t] \\
&b_{41} = -\tfrac{1}{3}c_1\exp[\tfrac{5}{3}c_1t] - Dc_1\exp[\tfrac{4}{3}c_1t]
\end{aligned}
\tag{2.8}
$$

where A, B, C and D are arbitrary constants.

Using equations (2.8) and (2.6) in equation (2.4), we finally get a four-parameter symmetry group for the force-free Duffing oscillator (2.2). The corresponding four vector fields are easily seen to be

$$
\begin{aligned}
&S_1 = y\frac{\partial}{\partial x} - [c_1y + x^3 + \tfrac{2}{9}c_1^2x]\frac{\partial}{\partial y} = X \\
&S_2 = \exp[\tfrac{1}{3}c_1t]\left\{[y + \tfrac{1}{3}c_1x]\frac{\partial}{\partial x} - [\tfrac{1}{3}c_1y + x^3 + \tfrac{1}{9}c_1^2x]\frac{\partial}{\partial y}\right\} \\
&S_3 = \exp[\tfrac{4}{3}c_1t][y^2 + \tfrac{2}{3}c_1xy + \tfrac{1}{2}x^4 + \tfrac{1}{9}c_1^2x^2]S_1 \\
&S_4 = \exp[\tfrac{4}{3}c_1t][y^2 + \tfrac{2}{3}c_1xy + \tfrac{1}{2}x^4 + \tfrac{1}{9}c_1^2x^2]S_2
\end{aligned}
\tag{2.9}
$$

where X is the dynamical vector field. It may be noted that for exactly the same parametric choice (2.7) the Painlevé property holds [13] for equation (2.1).

2.2. *Integral of motion*

The integral of motion associated with the vector fields (2.9) can be found using the procedure adopted by Sen and Tabor [8] for the Lorenz system. Since the vector fields S_3 and S_4 are not functionally independent we can use them to generate the integral of motion. We can easily check that the function $f(x, y, t)$ defined by

$$S_3 = f(x, y, t)S_1 \qquad S_4 = f(x, y, t)S_2 \tag{2.10a}$$

where

$$f = \exp[\tfrac{4}{3}c_1t][y^2 + \tfrac{2}{3}c_1xy + \tfrac{1}{2}x^4 + \tfrac{1}{9}c_1^2x^2] \tag{2.10b}$$

is an eigenfunction of the dynamical vector field,

$$X = y\frac{\partial}{\partial x} - [c_1y + x^3 + \tfrac{2}{9}c_1^2x]\frac{\partial}{\partial y}$$

that is

$$X(f) = -\tfrac{4}{3}c_1f. \tag{2.11}$$

The integral of motion I associated with a dynamical system satisfies the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial I}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 0. \tag{2.12}$$

Since

$$\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} = X$$

where X is the dynamical vector field, we can write the above equation as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = I_t + X(I) = 0. \tag{2.13}$$

Hence if we define

$$I = f = \exp[\tfrac{4}{3}c_1 t][y^2 + \tfrac{2}{3}c_1 xy + \tfrac{1}{2}x^4 + \tfrac{1}{9}c_1^2 x^2] \tag{2.14}$$

then it follows that $\mathrm{d}I/\mathrm{d}t = 0$. Thus the function f given in equation (2.10) turns out to be the integral of motion associated with the system (2.2).

We can use the integral of motion I associated with the dynamical system, equation (2.2), to find the explicit solution as follows. By using the transformations

$$W = (3/\sqrt{2}c_1)\exp(c_1 t/3) \qquad Z = -\sqrt{2}\exp(-c_1 t/3) \tag{2.15}$$

equation (2.1) for the parametric choice $2c_1^2 = 9c_2$ can be written as

$$W'' + W^3 = 0 \qquad ' = \mathrm{d}/\mathrm{d}z. \tag{2.16}$$

This can obviously be integrated as

$$W'^2 + \tfrac{1}{2}W^4 = \hat{I}. \tag{2.17}$$

Reverting to the old variables, it is easily seen that $\hat{I} = I$. Equation (2.17) can be integrated straightaway to give the Jacobian elliptic function solution [13]

$$x(t) = (\sqrt{2}c_1/3)\gamma\exp(-c_1 t/3)\mathrm{cn}(\gamma v; k) \qquad v = -\sqrt{2}\exp(-c_1 t/3) - Z_0 \tag{2.18}$$

where γ and Z_0 are arbitrary integration constants and $k^2 = \frac{1}{2}$.

2.3. *Infinite dimensional Lie algebra*

We can easily verify that the commutator algebra of the vector fields (2.9) is

$$\begin{aligned} &[S_1, S_2] = -\tfrac{1}{3}c_1 S_2 \qquad [S_1, S_3] = -\tfrac{4}{3}c_1 S_3 \\ &[S_1, S_4] = -\tfrac{5}{3}c_1 S_4 \qquad [S_2, S_3] = \tfrac{1}{3}c_1 S_4 \qquad [S_2, S_4] = 0 \end{aligned} \tag{2.19a}$$

and

$$[S_3, S_4] = -\tfrac{5}{3}c_1 S_5. \tag{2.19b}$$

Here the new vector field S_5 is defined as

$$S_5 = \{\exp[\tfrac{4}{3}c_1 t][y^2 + \tfrac{2}{3}c_1 xy + \tfrac{1}{2}x^4 + \tfrac{1}{9}c_1^2 x^2]\}^2 S_2 = I^2 S_2 \tag{2.20}$$

which is of fifth degree in the variable y.

Proceeding in a similar manner we can compute the commutator algebra including the new generator S_5. In this case we get the following relations in addition to the previous ones:

$$[S_5, S_1] = 3c_1 S_5 \qquad [S_5, S_2] = 0 \qquad [S_5, S_3] = 3c_1 S_6 \qquad [S_5, S_4] = 0 \tag{2.21}$$

where $S_6 = I^3 S_2$, which turns out to be a new generator.

Proceeding again, we obtain the new commutators as

$$\begin{aligned} &[S_6, S_1] = \tfrac{13}{3} c_1 S_6 \qquad [S_6, S_2] = 0 \\ &[S_6, S_3] = \tfrac{13}{3} c_1 S_7 \qquad [S_6, S_4] = 0 \qquad \text{and} \qquad [S_6, S_5] = 0 \end{aligned} \tag{2.22a}$$

where

$$S_7 = I^4 S_2. \tag{2.22b}$$

One can continue this process *ad infinitum*, introducing a new generator at each stage thereby leading to an infinite dimensional Lie algebra. Thus considering generators $[S_5, \ldots, S_m]$, where m is arbitrary, we find the commutator $[S_m, S_3]$ leads to a new generator S_{m+1}:

$$[S_m, S_3] = [I^{m-3} S_2, S_2] = \tfrac{1}{3}(4m - 11) c_1 I^{m-2} S_2. \tag{2.23}$$

We also note that this procedure does not include the generators $I^m S_1$, $m \geqslant 1$, which are also functionally dependent symmetry generators. Adjoining them with the previous set $S_1, S_2, \ldots, S_m$, we obtain the complete symmetry of degree m in the variable y for the equation (2.1).

Thus we find that only for the parametric choice $2c_1^2 = 9c_2$, the infinite dimensional Lie algebra of vector fields preserves the original flow, the consequence of which is that the system is completely integrable. In all other cases, one obtains only the trivial time translational symmetry and the system becomes non-integrable.

3. Holmes–Rand nonlinear oscillator

The Holmes–Rand nonlinear oscillator arises in certain flow induced structural vibration problems in which the structural nonlinearities act to maintain overall stability. In standard variables, it takes the form [14]

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0 \tag{3.1}$$

where α, β and γ are parameters. The system also has a close resemblance to the Duffing–van der Pol class of nonlinear oscillators [11].

3.1. Lie symmetries and integral of motion

Rewriting the above equation into a set of two first-order equations we get

$$\dot{x} = y$$
$$\dot{y} = -(\alpha + \beta x^2)y + \gamma x - x^3. \tag{3.2}$$

The invariance requirement of (3.2) under the infinitesimal transformations (A.2) can be written as

$$\dot{\eta}_1 = \eta_2$$
$$\dot{\eta}_2 = \eta_1(-2\beta xy + \gamma - 3x^2) - \eta_2(\alpha + \beta x^2). \tag{3.3}$$

As before, we make an ansatz for η_1 and η_2 so that they are polynomials in the variable y, for example quadratic to have non-trivial set of Lie vector fields:

$$\eta_1 = a_1 + a_2 y + a_3 y^2$$
$$\eta_2 = b_1 + b_2 y + b_3 y^2 \tag{3.4}$$

where the a_i's and b_i's, $i = 1, 2, 3$, are functions of t and x alone. Substituting this form in (3.3) and equating various powers of y we get the following set of linear partial differential equations:

$$a_{3x} = 0 \tag{3.5a}$$

$$b_{3x} + 2\beta x a_3 = 0 \tag{3.5b}$$

$$a_{2x} - 2(\alpha + \beta x^2)a_3 + a_{3t} - b_3 = 0$$
$$b_{2x} - (\alpha + \beta x^2)b_3 + b_{3t} - (\gamma - 3x^2)a_3 + 2\beta x a_2 = 0 \tag{3.5c}$$

$$a_{1x} - (\alpha + \beta x^2)a_2 + 2(\gamma x - x^3)a_3 + a_{2t} - b_2 = 0$$
$$b_{1x} + 2(\gamma x - x^3)b_3 + b_{2t} - (\gamma - 3x^2)a_2 + 2\beta x a_1 = 0 \tag{3.5d}$$

$$a_{1t} + (\gamma x - x^3)a_2 - b_1 = 0$$
$$b_{1t} + (\gamma x - x^3)b_2 - (\gamma - 3x^2)a_1 + (\alpha + \beta x^2)b_1 = 0. \tag{3.5e}$$

As described in the previous example, solving (3.5*a*)–(3.5*e*) consistently we obtain non-trivial forms for the functions a_1, a_2, a_3, b_1, b_2 and b_3 only for the parametric choice

$$\alpha = \frac{4}{\beta} \qquad \text{and} \qquad \gamma = -\left(\frac{3}{\beta^2}\right). \tag{3.6}$$

In this case we get a four-parameter symmetry group with the associated vector fields:

$$S_1 = y\frac{\partial}{\partial x} - \left[\left(\beta x^2 + \frac{4}{\beta}\right)y + x^3 + \frac{3}{\beta^2}x\right]\frac{\partial}{\partial y} = X$$

$$S_2 = \exp[(2/\beta)t]\left\{\left[y + \left(\frac{x}{\beta}\right)\right]\frac{\partial}{\partial x} - \left[\left(\beta x^2 + \frac{1}{\beta}\right)y + x^3 + \frac{x}{\beta^2}\right]\frac{\partial}{\partial y}\right\}$$

$$S_3 = \exp[(3/\beta)t]\left[y + \frac{\beta}{3}x^3 + \frac{x}{\beta}\right]S_1$$

$$S_4 = \exp[(3/\beta)t]\left[y + \frac{\beta}{3}x^3 + \frac{x}{\beta}\right]S_2 \tag{3.7}$$

where

$$X = y\frac{\partial}{\partial x} - \left[\left(\beta x^2 + \frac{4}{\beta}\right)y + x^3 + \frac{3}{\beta^2}x\right]\frac{\partial}{\partial y}$$

is now the dynamical vector field associated with (3.2).

As the vector fields S_3 and S_4 are not functionally independent we can use them to generate the integral of motion as in section 2.2 and we can conclude that the integral of motion for the equation (3.2) for the choice $\alpha = (4/\beta)$ and $\gamma = -(3/\beta^2)$ is

$$I = \exp[(3/\beta)t]\left[y + \frac{\beta}{3}x^3 + \frac{x}{\beta}\right]. \tag{3.8a}$$

Rewriting this as, we end up with a first-order inhomogeneous Abel's equation:

$$\dot{x} + \tfrac{1}{3}\beta x^3 + (1/\beta)x = I\exp[-(3/\beta)t]. \tag{3.8b}$$

Interestingly the Holmes–Rand nonlinear oscillator does not pass the Painlevé test as it admits a movable algebraic branch point (see equations (3.8)) and a local Laurent expansion in the form [11]

$$x(t) = \left(\frac{3}{2\beta}\right)\tau^{-(1/2)} + \left(\frac{3}{2\beta}\right)\left(\frac{3}{2\beta} - \frac{\alpha}{2}\right)\tau^{(1/2)} + a_3\tau + \cdots \tag{3.9}$$

where $\tau = (t - t_0)$ and t_0 and a_3 are arbitrary constants. However, it has already been pointed out that there exist second-order systems which are non-Painlevé but which nevertheless possess one integral of motion and hence are integrable [2, 15]. Our investigation shows that the choice (3.6) of the Holmes–Rand nonlinear oscillator belongs to the above category.

3.2. *Infinite dimensional Lie algebra*

The commutation relations between the four vector fields (3.7) are

$$\begin{aligned}
&[S_1, S_2] = -(2/\beta)S_2 \qquad [S_1, S_3] = -(3/\beta)S_3 \\
&[S_1, S_4] = -(5/\beta)S_4 \qquad [S_2, S_3] = (2/\beta)S_4 \qquad [S_2, S_4] = 0
\end{aligned} \tag{3.10a}$$

and

$$[S_3, S_4] = -(5/\beta)S_5 \tag{3.10b}$$

where

$$S_5 = \left\{\exp[(3/\beta)t]\left[y + \frac{\beta}{3}x^3 + \frac{x}{\beta}\right]\right\}^2 S_2$$

which is cubic power in the variable y.

Proceeding in a similar manner we can evaluate the commutators involving S_5, namely

$$[S_5, S_1] = (8/\beta)S_5 \qquad [S_5, S_2] = 0 \qquad [S_5, S_3] = (8/\beta)S_6 \qquad [S_5, S_4] = 0 \tag{3.11}$$

where $S_6 = I^3 S_2$ which turns out to be a new generator. As described in the previous section 2.3 we can continue this procedure *ad infinitum*. Thus considering generators $[S_5, \ldots, S_m]$, where m is arbitrary, we find that through the commutator

$$[S_3, S_m] = (1/\beta)(7 - 3m)I^{m-2}S_2 = S_{m+1} \tag{3.12}$$

a new generator is introduced.

4. Infinite dimensional Lie algebras of other dissipative systems

In this section we wish to point out that the other dynamical systems discussed in the literature also admit infinite dimensional Lie algebras. For this purpose, we have considered three physically important nonlinear dissipative systems namely the two-dimensional Lotka–Volterra equation, the three-wave interaction problem and the Lorenz system.

4.1. Two-dimensional Lotka–Volterra equation

The symmetry properties of the two-dimensional Lotka–Volterra model

$$\dot{x} = ax - xy \qquad \dot{y} = xy - by \tag{4.1}$$

have been discussed by Baumann and Freyberger recently [9]. By assuming the infinitesimals to be quadratic in y, they have found that equation (4.1) in a second-order version admits a three-parameter symmetry group provided the parametric condition, $a + b = 0$, is satisfied. The vector fields found by Baumann and Freyberger are

$$\begin{aligned} S_1 &= (ax - xy)\frac{\partial}{\partial x} + (ay + xy)\frac{\partial}{\partial y} \\ S_2 &= \exp[-at]\left[(xy)\frac{\partial}{\partial x} - (xy)\frac{\partial}{\partial y}\right] \\ S_3 &= \exp[-at](x + y)S_2 = IS_2 \end{aligned} \tag{4.2}$$

where $I = \exp[-at](x + y)$.

However, on closer examination, one also finds that a fourth vector field quadratic in y for (4.1) exists. Its form is

$$S_4 = \exp[-at]\left\{[x^2 - (1/a)x^2y - xy^2]\frac{\partial}{\partial x} + [(1/a)x^2y + 2xy + (1/a)xy^2 + y^2]\frac{\partial}{\partial y}\right\} \tag{4.3}$$

and this helps us to generate an infinite dimensional Lie algebras in the present case.

Now the commutator algebra between the vector fields S_1, S_2, S_3 and S_4 is

$$\begin{aligned} &[S_1, S_2] = aS_2 \qquad [S_1, S_3] = aS_3 \qquad [S_1, S_4] = 2aS_4 \\ &[S_2, S_3] = -aS_4 \qquad [S_2, S_4] = 0 \end{aligned} \tag{4.4a}$$

and

$$[S_3, S_4] = 2aS_5 \tag{4.4b}$$

where $S_5 = I^2 S_2$, which turns out to be a new vector field (cubic in the variable y). Now including the generator S_5 with the previous four generators and proceeding further, we can generate infinite sequence of symmetries and an infinite dimensional Lie algebra as described in the previous two sections.

4.2. Three-wave interaction problem

The Lie symmetries and their associated integrals of motion of the three-wave interaction problem have been studied by Almeida and Moreira [10]. In standard variables, the equation of motion can be written as

$$\begin{aligned}\dot{x} &= ax - by + z - 2y^2\\ \dot{y} &= ay + \tfrac{1}{2}ab + 2xy\\ \dot{z} &= -2z - 2xz\end{aligned}\tag{4.5}$$

where a and b are parameters. The above model is completely integrable for only one parametric choice, that is $a = -1$, $b = 0$, in the sense that here it admits two functionally independent integrals of motion and thereby the system can be reduced to a first-order equation which can be integrated by quadratures. Furthermore, (4.5) is also partially integrable for four other parametric choices where it admits only one integral of motion [10, 16].

For the completely integrable case the vector fields associated with (4.5) can be found (as in sections 2 and 3) to be (where the infinitesimals $\eta_i = 1, 2, 3$ are assumed to be up to quadratic in the variable z, which is sufficient in this case for a non-trivial set of Lie vector fields) [10]:

$$\begin{aligned}S_1 &= (z - x - 2y^2)\frac{\partial}{\partial x} + (2xy - y)\frac{\partial}{\partial y} - (2xz + 2z)\frac{\partial}{\partial z} = X\\ S_2 &= \mathrm{e}^t\left[(z - 2y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right]\\ S_3 &= \mathrm{e}^{2t}(z + x^2 + y^2)X = \mathrm{e}^{2t}(z + x^2 + y^2)S_1\\ S_4 &= yz\mathrm{e}^{3t}X = yz\mathrm{e}^{3t}S_1\\ S_5 &= \mathrm{e}^{2t}(z + x^2 + y^2)S_2\\ S_6 &= yz\mathrm{e}^{3t}S_2.\end{aligned}\tag{4.6}$$

Now constructing the commutator algebra between the above six vector fields, we get

$$\begin{aligned}&[S_1, S_2] = -S_2 \qquad [S_1, S_3] = -2S_3 \qquad [S_1, S_4] = -3S_4\\ &[S_1, S_5] = -3S_5 \qquad [S_1, S_6] = -4S_6 \qquad [S_2, S_3] = S_5\\ &[S_2, S_4] = S_6 \qquad [S_2, S_5] = 0 \qquad [S_2, S_6] = 0\\ &[S_3, S_4] = S_7 \qquad [S_3, S_5] = -3S_8 \qquad [S_3, S_6] = -4S_9\\ &[S_4, S_5] = -3S_9 \qquad [S_5, S_6] = 0\end{aligned}\tag{4.7}$$

where $S_7(= I_1 I_2 S_1)$, $S_8(= I_1^2 S_2)$ and $S_9(= I_1 I_2 S_2)$ are new vector fields which are cubic in the variable z. Thus in this case also we have generated three new vector fields without explicitly solving the invariance condition. Now including the new vector fields with the previous six vector fields and computing the commutator algebra between them again we can generate infinite sequence of symmetries from the basic vector fields S_1, S_2, S_3, S_4, S_5 and S_6.

Finally, for the case of partially integrable cases, equation (4.5) admits only two mutually commuting vector fields and no interesting Lie algebraic structures exist here.

4.3. Lorenz system

Recently Sen and Tabor have reported the Lie symmetries and the symmetry reductions of the Lorenz model [8]. The Lorenz equations are

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = -xz + rx - y \qquad \dot{z} = -xy - bz \tag{4.8}$$

where σ, r and b are real positive parameters. Sen and Tabor have shown that the equation (4.8) in its third-order version admits non-trivial Lie symmetries (where the infinitesimals η_i, $i = 1, 2, 3$ are assumed to be up to cubic in one of the variable y to bring out all the known integrable cases) for five parametric choices. Among them, one turns out to be a completely integrable case and the remaining four turn out to be partially integrable cases. For the completely integrable case, $\sigma = 1/2$, $b = 1$ and $r = 0$, the associated vector fields have been given as

$$\begin{aligned}
S_1 &= \tfrac{1}{2}(y - x)\frac{\partial}{\partial x} - (xz + y)\frac{\partial}{\partial y} - (xy - z)\frac{\partial}{\partial z} = X \\
S_2 &= \mathrm{e}^{t/2}\left[y\frac{\partial}{\partial x} - 2xz\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial z}\right] \\
S_3 &= \mathrm{e}^{2t}(x^4 - 2x^2z - y^2)S_2 \\
S_4 &= \mathrm{e}^{t}(x^2 - z)S_2.
\end{aligned} \tag{4.9}$$

However, on closer examination we have found that for this case of (4.8) there exist two more vector fields, say S_5 and S_6, in which the infinitesimals are cubic in y,

$$\begin{aligned}
S_5 &= \mathrm{e}^{2t}(x^4 - 2x^2z - y^2)S_1 \\
S_6 &= \mathrm{e}^{t}(x^2 - z)S_1.
\end{aligned} \tag{4.10}$$

Now constructing the commutator algebra between the six vector fields one can generate higher-order symmetries, say fourth power in the variable y in the infinitesimals η_i, $i = 1, 2, 3$ as in the previous cases. Now including the newly constructed vector field with the basic six vector fields and proceeding again as described in the previous sections one can generate the infinite sequence of symmetries and the associated infinite dimensional Lie algebra.

5. Conclusions

In this paper we have analysed the invariance and integrability properties of the force-free Duffing oscillator and Holmes–Rand nonlinear oscillator from the Lie symmetries point of view. We have found the integrable parameters along with the integrals of motion for the above two systems. We have also shown that the above two dynamical systems admit infinite dimensional Lie algebras in the integrable cases. We have further pointed out that the other dynamical systems, discussed in the literature, such as the two-dimensional Lotka–Volterra model, the three-wave interaction problem and the Lorenz system, also admit infinite dimensional Lie algebras for the completely integrable cases and finite dimensional Lie algebra for partially integrable cases. On the other hand, for the non-integrable systems one ends up with only a trivial time translational symmetry vector field alone, at least for the types of system we have considered here. Thus at the Lie algebra level, integrable and non-integrable systems, corresponding to two and three coupled first-order ordinary differential equations, seem to possess distinct characters which we expect to be true in higher dimensions as well. Further, the structure of the infinite dimensional Lie algebra of the different nonlinear dissipative systems we have studied here may apparently appear to be the same, but they are distinguished from each other in the form of structure constants and the form and number of multiplier functions I, I_1, I_2 etc (cf equations (2.23), (3.11), (4.4) and (4.7) and (2.14) and (3.8a)) in each one of the cases. Finally from the structure of the Lie algebra, it should be possible to generate the original vector fields also, when the basis vector fields are given. For example with specific structure constants, multiplier functions and commutation relations (cf equations (2.19) or (3.10) and (2.14) or (3.8a)) and solving them consistently with a basis of vector fields one can find the original vector fields, besides the integrals of motion. Thus, it appears that the Lie algebraic structure which has been brought out for the integrable nonlinear dissipative systems considered here contains much information about the flows, their integrable nature as well as the symmetry vector fields. It is clear that the invariance analysis of differential equations can bring out many interesting aspects in the study of nonlinear dissipative systems.

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Appendix A

Let us consider a set of first-order coupled nonlinear ordinary differential equations

$$\Delta_i\colon (x_j, \dot{x}_j) = 0 \qquad i = 1, 2, \ldots, N,\; j = 1, 2, \ldots, M. \tag{A.1}$$

Now we look for the invariance of the equation (A.1) under a one-parameter infinitesimal point transformations of the form

$$\begin{aligned} X_i &= x_i + \epsilon \eta_i(t, x_i) \qquad i = 1, 2, \ldots, M \\ T &= t + \epsilon \xi(t, x_i). \end{aligned} \tag{A.2}$$

The corresponding infinitesimal generator is

$$V = \xi(t, x_i)\frac{\partial}{\partial t} + \eta(t, x_i)\frac{\partial}{\partial x_i}. \tag{A.3}$$

We will take $\xi = 0$ without loss of generality. Then the evolutionary vector field takes the form

$$V = \eta_i \frac{\partial}{\partial x_i}. \tag{A.4}$$

Since we are interested in obtaining the Lie symmetries of the set of coupled ordinary differential equations, (A.1), which are of first order in nature, we must know the first prolongation of the vector field V (for more details and proof for this statement, see for example [17] p 106). The associated first extended operator is

$$\mathrm{Pr}^{(1)}\, V = \eta_i \frac{\partial}{\partial x_i} + \dot{\eta}_i \frac{\partial}{\partial \dot{x}_i} \tag{A.5}$$

where $\dot{\eta}_i = \mathrm{D}_t \eta_i$, $i = 1, 2$ and D_t is the total differential operator. An operator V is said to be the generator of a one-parameter symmetry group [17, 18] for (A.1) if, whenever (A.1) is satisfied and

$$\mathrm{Pr}^{(1)}\, V(\Delta_i)|_{\Delta_i=0} = \left[\eta_i \frac{\partial}{\partial x_i} + \dot{\eta}_i \frac{\partial}{\partial \dot{x}_i}\right](\Delta_i) = 0. \tag{A.6}$$

Substituting the specific equation of motion (A.1) in (A.6) and solving it consistently we get the Lie symmetries η_i.

We finally note that second or higher prolongations of the vector field V do not lead to any new symmetries for equations (A.1). For example, the second prolongation

$$\mathrm{Pr}^{(2)}\, V = \eta_i \frac{\partial}{\partial x_i} + \dot{\eta}_i \frac{\partial}{\partial \dot{x}_i} + \ddot{\eta}_i \frac{\partial}{\partial \ddot{x}_i} \tag{A.7}$$

where $\ddot{\eta}_i = \mathrm{d}^2\eta_i/\mathrm{d}t^2$, $i = 1, 2$, acting on the system (A.1) leads to the invariance condition

$$\mathrm{Pr}^{(2)}\, V(\Delta_i)|_{\Delta_i=0} = \left[\eta_i \frac{\partial}{\partial x_i} + \dot{\eta}_i \frac{\partial}{\partial \dot{x}_i}\right](\Delta_i) = 0 \tag{A.8}$$

which is the same as the right-hand side of (A.6). A similar result holds for higher prolongations too.

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